

# Stringy differential geometry, beyond Riemann

Imtak Jeon<sup>†</sup>, Kanghoon Lee<sup>‡</sup> and Jeong-Hyuck Park<sup>†</sup>

<sup>†</sup>*Department of Physics, Sogang University, Seoul 121-742, Korea*

<sup>‡</sup>*Center for Quantum Spacetime, Sogang University, Seoul 121-742, Korea*

imtak@sogang.ac.kr, kanghoon@sogang.ac.kr, park@sogang.ac.kr

While the fundamental object in Riemannian geometry is a metric, closed string theories call for us to put a two-form gauge field and a scalar dilaton on an equal footing with the metric. Here we propose a novel differential geometry which treats the three objects in a unified manner, manifests not only diffeomorphism and one-form gauge symmetry but also  $\mathbf{O}(D, D)$  T-duality, and enables us to rewrite the known low energy effective action of them as a single term. Further, we develop a corresponding vielbein formalism and gauge the internal symmetry which is given by a direct product of two local Lorentz groups,  $\mathbf{SO}(1, D-1) \times \overline{\mathbf{SO}}(1, D-1)$ . We comment that the notion of cosmological constant naturally changes.

PACS numbers: 04.60.Cf, 02.40.-k

## I. INTRODUCTION

Symmetry guides the structure of Lagrangians and organizes the physical laws into simple forms. For example, in Maxwell theory, the Abelian gauge symmetry does not allow for an explicit mass term of the vector potential, and Lorentz symmetry unifies the original Maxwell's four equations into two.

In general relativity, where the key quantity is the spacetime metric, the diffeomorphism symmetry first demands replacing ordinary derivatives by covariant derivatives which involve a connection. Setting the metric to be covariant constant determines the (torsionless) connection, *i.e.* the Christoffel symbol, in terms of the metric and its derivatives, and hence diffeomorphism uniquely picks up the scalar curvature as the covariant term which is lowest order in derivatives of the metric.

On the other hand, in string theories, the metric,  $g_{\mu\nu}$ , accompanies a two-form gauge field,  $B_{\mu\nu}$ , and a scalar dilation,  $\phi$ , since the three of them complete the bosonic massless sector of a closed string. Their low energy effective action is of the well-known form:

$$S_{\text{eff.}} = \int dx^D \sqrt{-g} e^{-2\phi} \left( R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right), \quad (1)$$

where  $R$  is the scalar curvature of the metric and  $H_{\lambda\mu\nu}$  is the three-form field strength of the two-form gauge field. Here and henceforth we consider an arbitrary spacetime dimension,  $D$ , without restricting ourselves to the critical values, 10 or 26. Each term in (1) is clearly invariant under the diffeomorphism as well as the one-form gauge symmetry,

$$x^\mu \rightarrow x^\mu + \delta x^\mu, \quad B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu. \quad (2)$$

Moreover, though not manifest, the action enjoys T-duality which mix the three companions,  $g_{\mu\nu}, B_{\mu\nu}, \phi$  in a nontrivial manner, first noted by Buscher [1–3] and further studied in [4–11]: If we redefine the dilaton,  $\phi \rightarrow d$ ,

and set a  $2D \times 2D$  symmetric matrix,  $\mathcal{H}_{AB}$  from  $g_{\mu\nu}, B_{\mu\nu}$  [12], as

$$e^{-2d} = \sqrt{-g} e^{-2\phi}, \quad \mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}, \quad (3)$$

T-duality is conveniently realized by an  $\mathbf{O}(D, D)$  rotation which acts on the  $2D$ -dimensional vector indices,  $A, B, \dots$ , in a standard manner, while  $d$  is taken to be an  $\mathbf{O}(D, D)$  singlet. The  $\mathbf{O}(D, D)$  group is defined by the invariance of the constant metric of the following form,

$$\mathcal{J}_{AB} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (4)$$

Throughout the present paper, this metric is being used to freely raise or lower the  $2D$ -dimensional vector indices. Indeed, Hull and Zwiebach [13, 14], later with Hohm [15, 16] (see also [17]), managed to rewrite the effective action (1) in terms of the redefined dilaton,  $d$ , the  $2D \times 2D$  matrix,  $\mathcal{H}_{AB}$ , and their ordinary derivatives, such that the  $\mathbf{O}(D, D)$  T-duality structure became manifest, yet the diffeomorphism and the one-form gauge symmetry were not any more. In their approach, the spacetime dimension is formally doubled from  $D$  to  $2D$ , with coordinates,  $x^\mu \rightarrow y^A = (\tilde{x}_\mu, x^\nu)$ . The new coordinates,  $\tilde{x}_\mu$ , may be viewed as the canonical conjugates of the winding modes of closed strings. However, as a field theory counterpart to the level matching condition in closed string theories, it is required that all the fields as well as all of their possible products should be annihilated by the  $\mathbf{O}(D, D)$  d'Alembert operator,  $\partial^2 = \partial_A \partial^A$ ,

$$\partial^2 \Phi \equiv 0, \quad \partial_A \Phi_1 \partial^A \Phi_2 \equiv 0. \quad (5)$$

This ‘level matching constraint’ – which we also assume in this paper – actually means that the theory is not

truly doubled: there is a choice of coordinates  $(\tilde{x}', x')$ , related to the original coordinates  $(\tilde{x}, x)$ , by an  $\mathbf{O}(D, D)$  rotation, in which all the fields do not depend on the  $\tilde{x}'$  coordinates [15]. Henceforth, the equivalence symbol, ‘ $\equiv$ ’, means an equality up to the constraint (5), or simply up to the winding coordinate independency, *i.e.*  $\frac{\partial}{\partial \tilde{x}_\mu} \equiv 0$ .

Combining the two types of the parameters,

$$X^A = (\Lambda_\mu, \delta x^\nu),$$

the diffeomorphism and the one-form gauge transformations (2) can be expressed in a unified fashion,

$$\begin{aligned} \delta_X \mathcal{H}_{AB} &\equiv X^C \partial_C \mathcal{H}_{AB} + 2\partial_{[A} X_{C]} \mathcal{H}^C{}_B + 2\partial_{[B} X_{C]} \mathcal{H}_A{}^C, \\ \delta_X (e^{-2d}) &\equiv \partial_A (X^A e^{-2d}). \end{aligned} \quad (6)$$

These expressions can be identified as the generalized Lie derivatives whose commutator leads to the Courant bracket [8, 11, 16, 18, 19]. In fact, in our previous work [20], starting from the observation that  $\mathcal{H}_{AB}$  given in (3) assumes a generic form of a symmetric  $\mathbf{O}(D, D)$  element [47], we constructed a certain differential operator which can be made compatible with the gauge transformations (6), being characterized by a projection:

$$P_{AB} = P_{BA} = \frac{1}{2}(\mathcal{J} + \mathcal{H})_{AB}, \quad P_A{}^B P_B{}^C = P_A{}^C. \quad (7)$$

In this work, generalizing the results of Ref.[20] (see also works by Siegel [7] and Hassan [21]), we propose a novel differential geometry apt for the unifying description of the closed string massless sector, which manifests all the relevant structures simultaneously:

- $\mathbf{O}(D, D)$  *T-duality*
- *Gauge symmetry*
  1. *Double-gauge symmetry*
    - *Diffeomorphism*
    - *One-form gauge symmetry*
  2. *Local Lorentz symmetry*

In particular, we reformulate the effective action (1) into a single term, like

$$S_{\text{eff.}} \equiv \int dy^{2D} e^{-2d} \mathcal{H}^{AB} S_{AB}. \quad (8)$$

## II. SEMI-COVARIANT DERIVATIVE

Employing the main idea of [20], we start with a differential operator,  $\nabla_C = \partial_C + \Gamma_C$ , which acts on a generic quantity carrying  $\mathbf{O}(D, D)$  vector indices,

$$\begin{aligned} \nabla_C T_{A_1 A_2 \dots A_n} &:= \partial_C T_{A_1 A_2 \dots A_n} - \omega \Gamma^B{}_{BC} T_{A_1 A_2 \dots A_n} \\ &\quad + \sum_{i=1}^n \Gamma_{CA_i}{}^B T_{A_1 \dots A_{i-1} B A_{i+1} \dots A_n}, \end{aligned} \quad (9)$$

where  $\omega$  denotes the given weight of each field,  $T_{A_1 A_2 \dots A_n}$ , and the connection must satisfy,

$$\Gamma_{CAB} + \Gamma_{CBA} = 0, \quad \Gamma_{ABC} + \Gamma_{CAB} + \Gamma_{BCA} = 0. \quad (10)$$

The only quantity which has a nontrivial weight in this paper is  $e^{-2d}$  having  $\omega = 1$ . Thanks to the symmetric properties (10), the ordinary derivatives in the definition of the generalized Lie derivative [8, 11, 16, 18, 19] can be replaced with our differential operator to give

$$\begin{aligned} \hat{\mathcal{L}}_X T_{A_1 \dots A_n} &:= X^B \nabla_B T_{A_1 \dots A_n} + \omega \nabla_B X^B T_{A_1 \dots A_n} \\ &\quad + \sum_{i=1}^n 2\nabla_{[A_i} X_{B]} T_{A_1 \dots A_{i-1} B A_{i+1} \dots A_n}, \end{aligned} \quad (11)$$

since the connection terms cancel.

We fix the connection by requiring

$$\begin{aligned} \nabla_A P_{BC} &= 0, \quad \nabla_A \bar{P}_{BC} = 0, \\ \nabla_A d &:= \partial_A d + \frac{1}{2} \Gamma^B{}_{BA} = 0, \end{aligned} \quad (12)$$

where  $\bar{P}_{AB} = (\mathcal{J} - P)_{AB}$  corresponds to the ‘anti-chiral’ projection which is complementary to the ‘chiral’ projection,  $P_{AB}$  in (7). Further,  $\nabla_A d$  is defined by the relation,

$$\nabla_A (e^{-2d}) = -2(\nabla_A d) e^{-2d}. \quad (13)$$

It follows that

$$\nabla_A \mathcal{J}_{BC} = 0, \quad \nabla_A \mathcal{H}_{BC} = 0. \quad (14)$$

That is to say, our differential operator thoroughly annihilates the closed string massless sector represented by  $d$  and  $\mathcal{H}_{AB}$ , which indicates that we are on a right track to achieve a unifying description of the massless modes [48].

In terms of  $P$ ,  $\bar{P}$ ,  $d$  and their derivatives, the connection reads explicitly (*cf.* [20]),

$$\begin{aligned} \Gamma_{CAB} &= 2(P \partial_C P \bar{P})_{[AB]} + 2(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E) \partial_D P_{EC} \\ &\quad - \frac{4}{D-1} (\bar{P}_{C[A} \bar{P}_{B]}{}^D + P_{C[A} P_{B]}{}^D) (\partial_D d + (P \partial^E P \bar{P})_{[ED]}). \end{aligned} \quad (15)$$

It is worth while to note that, similar to our previous cases [20, 22], the following derivative vanishes due to the level matching constraint (5),

$$P_I{}^A \bar{P}_J{}^B \Gamma^C{}_{AB} \partial_C \equiv 0. \quad (16)$$

Furthermore, if we set

$$\begin{aligned} \mathcal{P}_{CAB}{}^{DEF} &:= P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \frac{2}{D-1} P_{C[A} P_{B]}{}^{[E} P^{F]D}, \\ \bar{\mathcal{P}}_{CAB}{}^{DEF} &:= \bar{P}_C{}^D \bar{P}_{[A}{}^{[E} \bar{P}_{B]}{}^{F]} + \frac{2}{D-1} \bar{P}_{C[A} \bar{P}_{B]}{}^{[E} \bar{P}^{F]D}, \end{aligned} \quad (17)$$

which satisfy

$$\begin{aligned} \mathcal{P}_{CABDEF} &= \mathcal{P}_{DEFCAB} = \mathcal{P}_{C[AB]D[EF]}, \\ \mathcal{P}_{CAB}{}^{DEF} \mathcal{P}_{DEF}{}^{GHI} &= \mathcal{P}_{CAB}{}^{GHI}, \\ \mathcal{P}^A{}_{ABDEF} &= 0, \quad P^{AB} \mathcal{P}_{ABCDEF} = 0, \quad etc. \end{aligned} \quad (18)$$

the connection (15) belongs to the kernel of these rank six-projectors,

$$\mathcal{P}_{CAB}{}^{DEF} \Gamma_{DEF} = 0, \quad \bar{\mathcal{P}}_{CAB}{}^{DEF} \Gamma_{DEF} = 0. \quad (19)$$

In fact, the connection given in (15) is the unique solution to satisfy (10), (12) and (19).

Under the double-gauge transformations (6), the connection and the derivative (9) transform as

$$\begin{aligned} (\delta_X - \hat{\mathcal{L}}_X) \Gamma_{CAB} &\equiv 2[(\mathcal{P} + \bar{\mathcal{P}})_{CAB}{}^{FDE} - \delta_C^F \delta_A^D \delta_B^E] \partial_F \partial_{[D} X_{E]}, \\ (\delta_X - \hat{\mathcal{L}}_X) \nabla_C T_{A_1 \dots A_n} &\equiv \sum_i 2(\mathcal{P} + \bar{\mathcal{P}})_{CA_i}{}^{BFDE} \partial_F \partial_{[D} X_{E]} T_{\dots B \dots}. \end{aligned} \quad (20)$$

Hence, they are not double-gauge covariant. We say, a tensor is double-gauge covariant if and only if its double-gauge transformation agrees with the generalized Lie derivative. Nonetheless, the characteristic property of our derivative,  $\nabla_A$ , is that, combined with the projections, it can generate various  $\mathbf{O}(D, D)$  and double-gauge covariant quantities, as follows:

$$\begin{aligned} P_C^D \bar{P}_{A_1}{}^{B_1} \bar{P}_{A_2}{}^{B_2} \dots \bar{P}_{A_n}{}^{B_n} \nabla_D T_{B_1 B_2 \dots B_n}, \\ \bar{P}_C^D P_{A_1}{}^{B_1} P_{A_2}{}^{B_2} \dots P_{A_n}{}^{B_n} \nabla_D T_{B_1 B_2 \dots B_n}, \\ P^{AB} \bar{P}_{C_1}{}^{D_1} \bar{P}_{C_2}{}^{D_2} \dots \bar{P}_{C_n}{}^{D_n} \nabla_A T_{BD_1 D_2 \dots D_n}, \\ \bar{P}^{AB} P_{C_1}{}^{D_1} P_{C_2}{}^{D_2} \dots P_{C_n}{}^{D_n} \nabla_A T_{BD_1 D_2 \dots D_n}, \\ P^{AB} \bar{P}_{C_1}{}^{D_1} \bar{P}_{C_2}{}^{D_2} \dots \bar{P}_{C_n}{}^{D_n} \nabla_A \nabla_B T_{D_1 D_2 \dots D_n}, \\ \bar{P}^{AB} P_{C_1}{}^{D_1} P_{C_2}{}^{D_2} \dots P_{C_n}{}^{D_n} \nabla_A \nabla_B T_{D_1 D_2 \dots D_n}. \end{aligned} \quad (21)$$

Here, the latter second order derivatives actually follow from the recurrent applications of the former first order derivatives. The index  $n$  can be trivial, such that the covariant quantities include  $P^{AB} \nabla_A T_B$  and  $\bar{P}^{AB} \nabla_A T_B$ .

The above result suggests us to call the differential operator,  $\nabla_A$ , a ‘semi-covariant’ derivative.

### III. CURVATURES

Straightforward computation can show that, the usual curvature,

$$\mathcal{R}_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^E \Gamma_{BED} - \Gamma_{BC}{}^E \Gamma_{AED}, \quad (22)$$

set by the connection (15), is not double-gauge covariant, yet it satisfies

$$\mathcal{R}_{CDAB} = \mathcal{R}_{[CD][AB]}, \quad P_C^I \bar{P}_D^J \mathcal{R}_{IJAB} = 0. \quad (23)$$

We define, as for a key quantity in our formalism,

$$S_{ABCD} := \frac{1}{2} (\mathcal{R}_{ABCD} + \mathcal{R}_{CDAB} - \Gamma_{AB}^E \Gamma_{ECD}), \quad (24)$$

which can be shown, by brute force computation, to meet

$$\begin{aligned} S_{ABCD} &= S_{\{ABCD\}} := \frac{1}{2} (S_{[AB][CD]} + S_{[CD][AB]}), \\ S_{A[BCD]} &= 0, \\ P_I^A P_J^B \bar{P}_K^C \bar{P}_L^D S_{ABCD} &\equiv 0, \\ P_I^A \bar{P}_J^B P_K^C \bar{P}_L^D S_{ABCD} &\equiv 0, \\ P_I^A \bar{P}_J^C \mathcal{H}^{BD} S_{ABCD} &\equiv 0, \end{aligned} \quad (25)$$

and have a connection to a commutator,

$$P_I^A \bar{P}_J^B [\nabla_A, \nabla_B] T_C \equiv 2 P_I^A \bar{P}_J^B S_{CDAB} T^D. \quad (26)$$

Under the double-gauge transformations (6), we get

$$(\delta_X - \hat{\mathcal{L}}_X) S_{ABCD} \equiv 4 \nabla_{\{A} [(\mathcal{P} + \bar{\mathcal{P}})_{BCD\}}{}^{EFG} \partial_E \partial_{[F} X_{G]}, \quad (27)$$

from which double-gauge and  $\mathbf{O}(D, D)$  T-duality covariant, rank two-tensor as well as scalar follow,

$$P_I^A \bar{P}_J^B S_{AB}, \quad \mathcal{H}^{AB} S_{AB}. \quad (28)$$

Here we set

$$S_{AB} = S_{BA} := S^C{}_{ACB}, \quad (29)$$

which turns out to be, from direct computation, traceless,

$$S^A{}_A \equiv 0. \quad (30)$$

Especially, the covariant scalar constitutes the effective action (8) as

$$\mathcal{H}^{AB} S_{AB} \equiv R + 4 \square \phi - 4 \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu}, \quad (31)$$

and this is consistent with Refs.[16, 20].

Under arbitrary infinitesimal transformations of the dilaton and the projection (of which the latter should obey, from (7),  $\delta P = P \delta P \bar{P} + \bar{P} \delta P P$ ), we get

$$\delta S_{ABCD} = \nabla_{[A} \delta \Gamma_{B]CD} + \nabla_{[C} \delta \Gamma_{D]AB}, \quad (32)$$

where explicitly

$$\begin{aligned} \delta \Gamma_{CAB} &= 2 P_{[A}^D \bar{P}_{B]}^E \nabla_C \delta P_{DE} + 2 (\bar{P}_{[A}^D \bar{P}_{B]}^E - P_{[A}^D P_{B]}^E) \nabla_D \delta P_{EC} \\ &\quad - \frac{4}{D-1} (\bar{P}_{C[A} \bar{P}_{B]}^D + P_{C[A} P_{B]}^D) (\partial_D \delta d + P_{E[G} \nabla^G \delta P_{D]}^E) \\ &\quad - \Gamma_{FDE} \delta (\mathcal{P} + \bar{\mathcal{P}})_{CAB}{}^{FDE}. \end{aligned}$$

Now, with (32) and  $\nabla_A d = 0$ , from the manipulation,

$$\delta S_{\text{eff.}} \equiv \int dy^{2D} 2e^{-2d} (\delta P^{AB} S_{AB} - \delta d \mathcal{H}^{AB} S_{AB}),$$

it is very easy to rederive the equations of motion [16, 23]:

$$P_{(I}^A \bar{P}_{J)}^B S_{AB} = 0, \quad \mathcal{H}^{AB} S_{AB} = 0. \quad (33)$$

#### IV. DOUBLE-VIELBEIN

An interesting fact about  $\mathcal{J}_{AB}$  in (4) and  $\mathcal{H}_{AB}$  in (3) is that, they can be simultaneously diagonalized as (*c.f.* [16, 24, 25]),

$$\begin{aligned}\mathcal{J} &= \begin{pmatrix} V & \bar{V} \end{pmatrix} \begin{pmatrix} \eta^{-1} & 0 \\ 0 & -\bar{\eta} \end{pmatrix} \begin{pmatrix} V & \bar{V} \end{pmatrix}^t, \\ \mathcal{H} &= \begin{pmatrix} V & \bar{V} \end{pmatrix} \begin{pmatrix} \eta^{-1} & 0 \\ 0 & \bar{\eta} \end{pmatrix} \begin{pmatrix} V & \bar{V} \end{pmatrix}^t.\end{aligned}\quad (34)$$

Here  $\eta$  and  $\bar{\eta}$  are two copies of the  $D$ -dimensional Minkowskian metric. Both  $V$  and  $\bar{V}$  are  $2D \times D$  matrices which we name ‘double-vielbein’. They must satisfy

$$\begin{aligned}V &= PV, \quad V\eta^{-1}V^t = P, \quad V^t\mathcal{J}V = \eta, \quad V^t\mathcal{J}\bar{V} = 0, \\ \bar{V} &= \bar{P}\bar{V}, \quad \bar{V}\bar{\eta}\bar{V}^t = -\bar{P}, \quad \bar{V}^t\mathcal{J}\bar{V} = -\bar{\eta}^{-1},\end{aligned}\quad (35)$$

and hence they assume the following general form,

$$V_{Am} = \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})_m{}^\mu \\ (B+e)_{\nu m} \end{pmatrix}, \quad \bar{V}_A{}^{\bar{n}} = \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})^{\bar{n}\mu} \\ (B-\bar{e})_{\nu}{}^{\bar{n}} \end{pmatrix}.\quad (36)$$

Here  $e_\mu{}^m$  and  $\bar{e}_\nu{}^{\bar{n}}$  are two copies of the  $D$ -dimensional vielbein corresponding to the same spacetime metric (*c.f.* [7, 26, 27]),

$$e_\mu{}^m e_{\nu m} = \bar{e}_\mu{}^{\bar{n}} \bar{e}_{\nu \bar{n}} = g_{\mu\nu}. \quad (37)$$

We set  $B_{\mu m} = B_{\mu\nu}(e^{-1})_m{}^\nu$ ,  $B_{\mu\bar{n}} = B_{\mu\nu}(\bar{e}^{-1})^{\bar{n}\nu}$ , *etc.* We may identify  $(B+e)_{\mu m}$  and  $(B-\bar{e})_{\mu\bar{n}}$  as two copies of the vielbein for the winding mode coordinate,  $\tilde{x}_\mu$ , as

$$(B+e)_\mu{}^m (B+e)_{\nu m} = (B-\bar{e})_\mu{}^{\bar{n}} (B-\bar{e})_{\nu \bar{n}} = (g-Bg^{-1}B)_{\mu\nu}. \quad (38)$$

The whole internal symmetry group is, from (34), given by a direct product of a pair of  $D$ -dimensional Lorentz groups, *i.e.*  $\mathbf{SO}(1, D-1) \times \overline{\mathbf{SO}}(1, D-1)$ , of which the former and the latter respectively acts on each unbarred and barred small Roman alphabet index, while the  $\mathbf{O}(D, D)$  T-duality group acts only on the capital indices. Indeed, if we parametrize a  $2D \times 2D$  skew-symmetric matrix as

$$h_{AB} = -h_{BA} = \begin{pmatrix} \alpha^{\mu\sigma} & -(\beta^t)^\mu{}_\rho \\ \beta_\nu{}^\sigma & \gamma_{\nu\rho} \end{pmatrix} = \begin{pmatrix} -\alpha^{\sigma\mu} & -\beta_\rho{}^\mu \\ \beta_\nu{}^\sigma & -\gamma_{\rho\nu} \end{pmatrix}, \quad (39)$$

the vectorial  $\mathbf{so}(D, D)$  transformations of the double-vielbein (36),

$$\hat{\delta}_h V_A = h_A{}^B V_B, \quad \hat{\delta}_h \bar{V}_A = h_A{}^B \bar{V}_B, \quad (40)$$

give well-defined  $\mathbf{so}(D, D)$  transformation rules for the pair of vielbeins and also the two-form field,

$$\begin{aligned}\hat{\delta}_h e_{\mu m} &= (\beta_\mu{}^\nu - B_{\mu\rho}\alpha^{\rho\nu} + g_{\mu\rho}\alpha^{\rho\nu})e_{\nu m}, \\ \hat{\delta}_h \bar{e}_{\mu\bar{n}} &= (\beta_\mu{}^\nu - B_{\mu\rho}\alpha^{\rho\nu} - g_{\mu\rho}\alpha^{\rho\nu})\bar{e}_{\nu\bar{n}}, \\ \hat{\delta}_h B_{\mu\nu} &= \gamma_{\mu\nu} - 2\beta_{[\mu}{}^\rho B_{\nu]\rho} - g_{\mu\rho}\alpha^{\rho\sigma}g_{\sigma\nu} - B_{\mu\rho}\alpha^{\rho\sigma}B_{\sigma\nu},\end{aligned}\quad (41)$$

where we put,  $\hat{\delta}_h := \delta_h - y^A h_A{}^B \partial_B$ , for our short hand notation. Both  $\hat{\delta}_h e_{\mu m}$  and  $\hat{\delta}_h \bar{e}_{\mu\bar{n}}$ , despite of a particular sign difference therein (41), give the same transformation rule for the metric,

$$\hat{\delta}_h g_{\mu\nu} = (\beta g + g\beta^t - g\alpha B - B\alpha g)_{\mu\nu}, \quad (42)$$

which agrees with  $\hat{\delta}_h \mathcal{H}_{AB} = h_A{}^C \mathcal{H}_{CB} + h_B{}^C \mathcal{H}_{AC}$ . In fact, the terms of the sign difference in  $\hat{\delta}_h e$  and  $\hat{\delta}_h \bar{e}$  (41) can be identified as local Lorentz transformations, such as  $g_{\mu\rho}\alpha^{\rho\nu}e_{\nu m} = e_\mu{}^n \alpha_{nm}$ ,  $g_{\mu\rho}\alpha^{\rho\nu}\bar{e}_{\nu\bar{m}} = \bar{e}_\mu{}^{\bar{n}} \alpha_{\bar{n}\bar{m}}$ . Hence, they do not contribute to the variation of the metric,  $\hat{\delta}_h g_{\mu\nu}$  (42), and the  $\mathbf{so}(D, D)$  transformation of  $(\bar{e}^{-1}e)^{\bar{m}}{}_n$  matches local Lorentz transformations,

$$\hat{\delta}_h (\bar{e}^{-1}e)^{\bar{m}}{}_n = 2\alpha^{\bar{m}}{}_{\bar{k}}(\bar{e}^{-1}e)^{\bar{k}}{}_n = 2(\bar{e}^{-1}e)^{\bar{m}}{}_{\bar{k}}\alpha^{\bar{k}}{}_n. \quad (43)$$

Furthermore, direct computation can show that,  $V_{Am}$  and  $\bar{V}_A{}^{\bar{n}}$  are double-gauge covariant vectors, as their diffeomorphism plus one-form gauge symmetry transformations (2) coincide with their generalized Lie derivatives,

$$\begin{aligned}\delta_X V_{Am} &\equiv X^B \partial_B V_{Am} + 2\partial_{[A} X_{B]} V^B{}_m = \hat{\mathcal{L}}_X V_{Am}, \\ \delta_X \bar{V}_A{}^{\bar{n}} &\equiv X^B \partial_B \bar{V}_A{}^{\bar{n}} + 2\partial_{[A} X_{B]} \bar{V}^{B\bar{n}} = \hat{\mathcal{L}}_X \bar{V}_A{}^{\bar{n}}.\end{aligned}\quad (44)$$

We also define ‘twins’ of the double-vielbein, by exchanging  $e_{\mu m}$  and  $\bar{e}_{\mu\bar{m}}$  in (36),

$$U_{A\bar{n}} := \frac{1}{\sqrt{2}} \begin{pmatrix} (\bar{e}^{-1})^{\bar{n}\mu} \\ (B+\bar{e})_{\nu\bar{n}} \end{pmatrix}, \quad \bar{U}_A{}^m := \frac{1}{\sqrt{2}} \begin{pmatrix} (e^{-1})^{m\mu} \\ (B-e)_{\nu}{}^m \end{pmatrix}. \quad (45)$$

They are conjugate to the double-vielbein, satisfying the identical properties (35), with  $U \leftrightarrow V$ ,  $\bar{U} \leftrightarrow \bar{V}$ , and  $\eta \leftrightarrow \bar{\eta}$ . Also they are double-gauge covariant as in (44), and apparently local Lorentz covariant. But, due to the predetermined transformation rules of the double-vielbein, (41), the twins are not qualified as  $\mathbf{O}(D, D)$  covariant vectors like (40). Rather they transform as

$$\begin{aligned}\hat{\delta}_h U_{A\bar{n}} &= h_A{}^B U_{B\bar{n}} - 2U_{A\bar{m}}\alpha^{\bar{m}}{}_{\bar{n}}, \\ \hat{\delta}_h \bar{U}_A{}^m &= h_A{}^B \bar{U}_B{}^m + 2\bar{U}_{Am}\alpha^m{}_n.\end{aligned}\quad (46)$$

A crucial ability of the double-vielbein (36) and its twin (45) is that, they can pull back the chiral and the anti-chiral  $2D$  indices to the more familiar  $D$ -dimensional

ones, without losing any information since it is, from (35), an invertible process. We pull back the double-gauge covariant rank two-tensor in (33), to obtain

$$2S_{AB}V^A{}_m\bar{V}^B{}_{\bar{n}} \equiv R_{m\bar{n}} + 2D_m D_{\bar{n}}\phi - \frac{1}{4}H_{m\mu\nu}H_{\bar{n}}{}^{\mu\nu} - (\partial^\lambda\phi)H_{\lambda m\bar{n}} + \frac{1}{2}D^\lambda H_{\lambda m\bar{n}}. \quad (47)$$

Here we put  $D_m = (e^{-1})_m{}^\mu D_\mu$ ,  $D_{\bar{n}} = (\bar{e}^{-1})_{\bar{n}}{}^\mu D_\mu$ , and set  $D_\mu$  to be a  $D$ -dimensional differential operator which is covariant with respect to the diffeomorphism and the pair of local Lorentz groups, such that  $D_\lambda g_{\mu\nu} = 0$ ,  $D_\mu e_{\nu m} = 0$ ,  $D_\mu \bar{e}_{\nu \bar{n}} = 0$ . With the standard diffeomorphism covariant derivative,  $\nabla_\mu$ , which involves the Christoffel symbol, and local Lorentz connections,  $\omega_{\mu mn} = (e^{-1})_m{}^\nu \nabla_\mu e_{\nu n}$ ,  $\bar{\omega}_{\mu \bar{m}\bar{n}} = (\bar{e}^{-1})_{\bar{m}}{}^\nu \nabla_\mu \bar{e}_{\nu \bar{n}}$ , we have

$$D_\mu = \nabla_\mu + \omega_\mu + \bar{\omega}_\mu. \quad (48)$$

Similarly, if we compute  $S_{AB}V^A{}_m\bar{U}^B{}_{\bar{n}}$ , the resulting expression is essentially identical to (47), with the removal of the bar symbol from the subscript index,  $\bar{n}$ . Then, as expected from (33), the symmetric and the anti-symmetric parts of it correspond to the equations of motion of the effective action (1) for  $g_{\mu\nu}$  and  $B_{\mu\nu}$  respectively.

## V. GAUGING THE INTERNAL SYMMETRY

We now consider gauging the internal symmetry and introduce a corresponding generalized derivative,  $\mathcal{D}_A$ , which is semi-covariant for the double-gauge symmetry and fully covariant for the pair of local Lorentz groups. Schematically, we write

$$\mathcal{D}_A = \nabla_A + \Omega_A + \bar{\Omega}_A, \quad (49)$$

where  $\Omega_A$  and  $\bar{\Omega}_A$  are the connections for the local Lorentz symmetry,  $\mathbf{SO}(1, D-1)$  and  $\overline{\mathbf{SO}}(1, D-1)$ , respectively. As the analyses are parallel, we first focus on a single local Lorentz group,  $\mathbf{SO}(1, D-1)$ . In order to relate the connection,  $\Omega_{Amn}$ , to the double-vielbein and its twin, specifically  $V_{An}$  and  $\bar{U}_{An}$ , we propose to impose,

$$\begin{aligned} \bar{P}_A{}^B \mathcal{D}_B V_{Cm} &= \bar{P}_A{}^B (\nabla_B V_{Cm} + \Omega_{Bmn} V_C{}^n) = 0, \\ P_A{}^B \mathcal{D}_B \bar{U}_{Cm} &= P_A{}^B (\nabla_B \bar{U}_{Cm} + \Omega_{Bmn} \bar{U}_C{}^n) = 0. \end{aligned} \quad (50)$$

These equations are not only local Lorentz covariant but also, from (21), double-gauge covariant. As a unique solution to them, the connection is obtained,

$$\Omega_{Amn} = \bar{P}_A{}^B V_{Cm} \nabla_B V_C{}^n - P_A{}^B \bar{U}_{Cm} \nabla_B \bar{U}_C{}^n. \quad (51)$$

Indeed, one can check straightforwardly that, the right hand side of (51) is a double-gauge covariant vector, and

that it transforms properly under the local Lorentz transformation. Similarly for  $\overline{\mathbf{SO}}(1, D-1)$ , we get

$$\begin{aligned} \bar{\Omega}_{A\bar{m}\bar{n}} &= \bar{P}_A{}^B U_{C\bar{m}} \nabla_B U^C{}_{\bar{n}} - P_A{}^B \bar{V}_{C\bar{m}} \nabla_B \bar{V}^C{}_{\bar{n}} \\ &= [(\bar{e}^{-1}e)(\Omega_A + \partial_A)(e^{-1}\bar{e})]_{\bar{m}\bar{n}}. \end{aligned} \quad (52)$$

Upon the level matching constraint, they reduce to

$$\Omega_A \equiv \begin{pmatrix} -\frac{1}{2}H^\mu \\ \omega_\nu - \frac{1}{2}B_{\nu\rho}H^\rho \end{pmatrix}, \quad \bar{\Omega}_A \equiv \begin{pmatrix} -\frac{1}{2}\bar{H}^\mu \\ \bar{\omega}_\nu - \frac{1}{2}B_{\nu\rho}\bar{H}^\rho \end{pmatrix}, \quad (53)$$

where we let  $(H_\mu)_{mn} = H_{\mu mn}$ ,  $(\bar{H}_\mu)_{\bar{m}\bar{n}} = H_{\mu \bar{m}\bar{n}}$ , etc.

Now, following [22], with the semi-covariant derivative,  $\nabla_A$ , we define a modified Cartan-Maurer curvature,

$$\mathcal{F}_{AB} := \nabla_A \Omega_B - \nabla_B \Omega_A + [\Omega_A, \Omega_B]. \quad (54)$$

After projecting its two  $\mathbf{O}(D, D)$  vector indices into opposite chiralities, we acquire a rank four-tensor,

$$\begin{aligned} \mathcal{C}_{mnpq} &:= 2(\mathcal{F}_{AB})_{mn} V^A{}_p \bar{U}^B{}_q \\ &\equiv R_{mnpq} + D_{(p} H_{q)mn} - \frac{1}{4} H_{mn}{}^l H_{pql} - \frac{3}{4} H_{m[n}{}^l H_{pq]l}, \end{aligned} \quad (55)$$

which is gauge, *i.e.* double-gauge plus local Lorentz, covariant. However, since the twin of the double-vielbein is not an  $\mathbf{O}(D, D)$  covariant vector (46), the connection (51) and consequently the rank four-tensor (55) are not fully  $\mathbf{O}(D, D)$  covariant. Indeed, straightforward computation can show

$$\hat{\delta}_h \mathcal{C}_{mnpq} = 2\mathcal{C}_{mnpq} \alpha^r{}_q - 4V^A{}_p \bar{U}^B{}_q \mathcal{D}_B \mathcal{D}_A \alpha_{mn}. \quad (56)$$

Turning off the  $\mathbf{O}(D, D)$  parameter,  $\alpha^{\mu\nu}$ , in (39) breaks  $\mathbf{O}(D, D)$  to  $\mathbf{O}(D) \rtimes \mathbf{GL}(D)$ , as the remaining ones,  $\beta_\mu{}^\nu$  and  $\gamma_{\mu\nu}$ , form  $\mathbf{gl}(D)$  and  $\mathbf{so}(D)$ , respectively. Thus, the rank four-tensor (55) is not  $\mathbf{O}(D, D)$  but  $\mathbf{O}(D) \rtimes \mathbf{GL}(D)$  covariant.

Since  $\bar{\Omega}_A$  is linked to  $\Omega_A$  by a gauge transformation-like relation (52), the corresponding rank four-tensor to the other local Lorentz group,  $\overline{\mathbf{SO}}(1, D-1)$ , is essentially identical to (55), replacing the unbarred indices to the barred indices,  $\mathcal{C}_{mnpq} \rightarrow \mathcal{C}_{\bar{m}\bar{n}\bar{p}\bar{q}}$ .

Alternatively, we may consider gauging only the diagonal subgroup of  $\mathbf{SO}(1, D-1) \times \overline{\mathbf{SO}}(1, D-1)$ , *i.e.* a single Lorentz group acting on both the barred and the unbarred small Roman alphabet indices simultaneously. Then, the corresponding connection,  $\Omega'_A = \bar{P}_A{}^B V_C \nabla_B V^C - P_A{}^B \bar{V}_C \nabla_B \bar{V}^C$ , as well as its rank-four tensor are all fully  $\mathbf{O}(D, D)$  covariant. Choosing the gauge,  $e_{\mu m} \equiv \bar{e}_{\mu \bar{m}}$ , spontaneously breaks the T-duality group,  $\mathbf{O}(D, D)$  to  $\mathbf{O}(D) \rtimes \mathbf{GL}(D)$ , and the internal symmetry,  $\mathbf{SO}(1, D-1) \times \overline{\mathbf{SO}}(1, D-1)$  to the diagonal subgroup. Further, it reduces the fully  $\mathbf{O}(D, D)$  covariant rank four-tensor to  $\mathcal{C}_{mnpq}$  in (55). In this way,  $\mathcal{C}_{mnpq}$



can be identified as a gauge fixed version of the fully  $\mathbf{O}(D, D)$  covariant rank four-tensor.

Finally, we pull back the covariant derivatives in (21),

$$\begin{aligned}
V^A{}_l \mathcal{D}_A T_{\bar{k}_1 \bar{k}_2 \dots \bar{k}_n} &\equiv \frac{1}{\sqrt{2}} \hat{D}_l T_{\bar{k}_1 \bar{k}_2 \dots \bar{k}_n}, \\
\bar{V}^A{}_{\bar{l}} \mathcal{D}_A T_{k_1 k_2 \dots k_n} &\equiv \frac{1}{\sqrt{2}} \hat{D}_{\bar{l}} T_{k_1 k_2 \dots k_n}, \\
P^{AB} \mathcal{D}_A T_{B \bar{k}_1 \dots \bar{k}_n} &\equiv \frac{1}{\sqrt{2}} \hat{D}^l T_{l \bar{k}_1 \dots \bar{k}_n} - \sqrt{2} \hat{D}^l \phi T_{l \bar{k}_1 \dots \bar{k}_n}, \\
\bar{P}^{AB} \mathcal{D}_A T_{B k_1 \dots k_n} &\equiv -\frac{1}{\sqrt{2}} \hat{D}^{\bar{l}} T_{\bar{l} k_1 \dots k_n} + \sqrt{2} \hat{D}^{\bar{l}} \phi T_{\bar{l} k_1 \dots k_n}, \\
P^{AB} \mathcal{D}_A \mathcal{D}_B T_{\bar{k}_1 \dots \bar{k}_n} &\equiv \frac{1}{2} \hat{D}^\mu \hat{D}_\mu T_{\bar{k}_1 \dots \bar{k}_n} - \hat{D}^\mu \phi \hat{D}_\mu T_{\bar{k}_1 \dots \bar{k}_n}, \\
\bar{P}^{AB} \mathcal{D}_A \mathcal{D}_B T_{k_1 \dots k_n} &\equiv -\frac{1}{2} \hat{D}^\mu \hat{D}_\mu T_{k_1 \dots k_n} + \hat{D}^\mu \phi \hat{D}_\mu T_{k_1 \dots k_n},
\end{aligned} \tag{57}$$

where we put, as for  $D$ -dimensional tensors which are  $\mathbf{O}(D, D)$  singlets,

$$T_{l_1 \dots l_m \bar{k}_1 \dots \bar{k}_n} = T_{A_1 \dots A_m B_1 \dots B_n} V^{A_1}{}_{l_1} \dots V^{A_m}{}_{l_m} \bar{V}^{B_1}{}_{\bar{k}_1} \dots \bar{V}^{B_n}{}_{\bar{k}_n},$$

and, instead of  $D_\mu$  in (48), we set, for  $\hat{D}_m = (e^{-1})_m{}^\mu \hat{D}_\mu$  and  $\hat{D}_{\bar{n}} = (\bar{e}^{-1})_{\bar{n}}{}^\mu \hat{D}_\mu$ ,

$$\hat{D}_\mu := \nabla_\mu + (\omega_\mu + \frac{1}{2} H_\mu) + (\bar{\omega}_\mu - \frac{1}{2} \bar{H}_\mu). \tag{58}$$

## VI. DISCUSSION

Since the internal symmetry is given by the direct product of two  $D$ -dimensional Lorentz groups, the corresponding gamma matrices are a pair of  $D$ -dimensional sets, rather than a single  $2D$ -dimensional set. Hence the size of spinors will not increase exponentially, from  $2^{D/2}$  to  $2^D$ , but will remain the same. This seems to be a desired property while attempting to supersymmetrize our formalism, towards a unifying description of the type IIA and IIB supergravities, *e.g.* [21, 26, 27].

Our results, especially the rank four-tensor (55), may provide powerful toolkits to organize, in a simple fashion, the higher order derivative corrections to the stringy

effective actions where the four-index Riemann or Weyl tensors are known to play a crucial role [28–34].

Application to doubled sigma models [35–38], connection to generalized geometry [39, 40], or generalization to  $\mathcal{M}$ -theory [41, 42] are also of interest.

In the stringy differential geometry we have proposed, the dilaton,  $d$ , appears only explicitly as the overall factor of the action, and its derivatives are completely absorbed into the connection (15), which therefore implies the tight symmetric structure of our formalism. Furthermore, it appears that the natural “cosmological constant term” is nothing but

$$\int dy^{2D} e^{-2d} \Lambda \equiv \int dx^D \sqrt{-g} e^{-2\phi} \Lambda, \tag{59}$$

since  $e^{-2d}$  is the only  $\mathbf{O}(D, D)$  singlet, scalar density with the weight of unity, providing the ‘volume form’ for the double field theory action. As  $\phi$  dynamically grows, this term becomes exponentially suppressed, irrespective of the choice of the frame, *i.e.* string or Einstein (see *e.g.* [43–46]). In this way, the notion of the cosmological constant naturally changes in our stringy differential geometry. This might provide a new spin on the cosmological constant problem.

It has been said that string theory is a piece of 21st century physics that happened to fall into the 20th century [49]. Perhaps, our formalism might provide some clue to a new framework beyond Riemannian geometry.

## Acknowledgements

We wish to thank Neil Copland and Michael Green for helpful comments. The work was supported by the National Research Foundation of Korea (NRF) grants funded by the Korea government (MEST) with the grant numbers, 2005-0049409 (CQUeST) and 2010-0002980.

- 
- [1] T. H. Buscher, Phys. Lett. B **159** (1985) 127.
  - [2] T. H. Buscher, Phys. Lett. B **194** (1987) 59.
  - [3] T. H. Buscher, Phys. Lett. B **201** (1988) 466.
  - [4] A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. B **322** (1989) 167.
  - [5] A. A. Tseytlin, Phys. Lett. B **242**, 163 (1990).
  - [6] A. A. Tseytlin, Nucl. Phys. B **350**, 395 (1991).
  - [7] W. Siegel, Phys. Rev. D **47**, 5453 (1993) [arXiv:hep-th/9302036].
  - [8] W. Siegel, Phys. Rev. D **48**, 2826 (1993) [arXiv:hep-th/9305073].
  - [9] E. Alvarez, L. Alvarez-Gaume and Y. Lozano, Phys. Lett. B **336** (1994) 183 [arXiv:hep-th/9406206].
  - [10] A. Giveon, M. Porrati and E. Rabinovici, Phys. Rept. **244**, 77 (1994) [arXiv:hep-th/9401139].
  - [11] M. Grana, R. Minasian, M. Petrini and D. Waldram, JHEP **0904** (2009) 075 [arXiv:0807.4527 [hep-th]].
  - [12] M. J. Duff, Nucl. Phys. B **335** (1990) 610.
  - [13] C. Hull and B. Zwiebach, JHEP **0909**, 099 (2009) [arXiv:0904.4664 [hep-th]].
  - [14] C. Hull and B. Zwiebach, JHEP **0909**, 090 (2009) [arXiv:0908.1792 [hep-th]].
  - [15] O. Hohm, C. Hull and B. Zwiebach, JHEP **1007**, 016 (2010) [arXiv:1003.5027 [hep-th]].
  - [16] O. Hohm, C. Hull and B. Zwiebach, JHEP **1008**, 008 (2010) [arXiv:1006.4823 [hep-th]].
  - [17] O. Hohm, JHEP **1104**, 103 (2011) [arXiv:1103.0032 [hep-th]].
  - [18] T. Courant, Dirac Manifolds, Trans. Amer. Math. Soc. **319**: 631–661, 1990.

- [19] M. Gualtieri, Ph.D. Thesis “Generalized complex geometry,” arXiv:math/0401221.
- [20] I. Jeon, K. Lee and J.-H. Park, JHEP **1104** (2011) 014 [arXiv:1011.1324 [hep-th]].
- [21] S. F. Hassan, Nucl. Phys. B **454** (1995) 86 [arXiv:hep-th/9408060].
- [22] I. Jeon, K. Lee and J.-H. Park, Phys. Lett. B (2011), doi:10.1016/j.physletb.2011.05.051 [arXiv:1102.0419 [hep-th]].
- [23] S. K. Kwak, JHEP **1010** (2010) 047 [arXiv:1008.2746 [hep-th]].
- [24] O. Hohm and S. K. Kwak, J. Phys. A **44** (2011) 085404 [arXiv:1011.4101 [hep-th]].
- [25] O. Hohm and S. K. Kwak, arXiv:1103.2136 [hep-th].
- [26] S. F. Hassan, Nucl. Phys. B **568** (2000) 145 [arXiv:hep-th/9907152].
- [27] S. F. Hassan, Nucl. Phys. B **583** (2000) 431 [arXiv:hep-th/9912236].
- [28] M. B. Green and J. H. Schwarz, Phys. Lett. B **149** (1984) 117.
- [29] D. J. Gross and E. Witten, Nucl. Phys. B **277** (1986) 1.
- [30] M. T. Grisaru, D. Zanon, Phys. Lett. B **177** (1986) 347.
- [31] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B **293** (1987) 385.
- [32] E. Bergshoeff, I. Entrop and R. Kallosh, Phys. Rev. D **49** (1994) 6663 [arXiv:hep-th/9401025].
- [33] K. A. Meissner, Phys. Lett. B **392** (1997) 298 [arXiv:hep-th/9610131].
- [34] K. Peeters, P. Vanhove and A. Westerberg, Class. Quant. Grav. **18** (2001) 843 [arXiv:hep-th/0010167].
- [35] C. M. Hull, JHEP **0510**, 065 (2005) [arXiv:hep-th/0406102].
- [36] C. M. Hull, JHEP **0707**, 080 (2007) [arXiv:hep-th/0605149].
- [37] D. S. Berman, N. B. Copland and D. C. Thompson, Nucl. Phys. B **791** (2008) 175 [arXiv:0708.2267 [hep-th]].
- [38] D. S. Berman and D. C. Thompson, Phys. Lett. B **662** (2008) 279 [arXiv:0712.1121 [hep-th]].
- [39] N. Hitchin, Quart. J. Math. Oxford Ser. **54**, 281 (2003) [arXiv:math/0209099].
- [40] N. Hitchin, arXiv:1008.0973 [math.DG].
- [41] D. S. Berman and M. J. Perry, arXiv:1008.1763 [hep-th].
- [42] D. S. Berman, H. Godazgar and M. J. Perry, arXiv:1103.5733 [hep-th].
- [43] J. J. Halliwell, Phys. Lett. B **185** (1987) 341.
- [44] A. A. Tseytlin, Phys. Rev. Lett. **66** (1991) 545.
- [45] K. A. Meissner and G. Veneziano, Mod. Phys. Lett. A **6** (1991) 3397 [arXiv:hep-th/9110004].
- [46] J. E. Lidsey, D. Wands and E. J. Copeland, Phys. Rept. **337** (2000) 343 [arXiv:hep-th/9909061].
- [47] The expression of  $\mathcal{H}_{AB}$  in (3) is the most general form of a  $2D \times 2D$  matrix satisfying,  $\mathcal{H}_{AB} = \mathcal{H}_{BA}$ ,  $\mathcal{H}_A{}^B \mathcal{H}_B{}^C = \delta_A{}^C$ , and that the upper left  $D \times D$  block is non-degenerate [20].
- [48] This differs from our previous work [20] where only  $\mathcal{H}_{AB}$  was annihilated and the dilaton was treated separately.
- [49] *Anonymous Italian*.